(2) It also says that if 2 eigenforms f, g have the same eigenvalues $\{\lambda_m\}$ and they're both normalized to have first F-coef as 1 then $f \equiv g$.
(This is called a multiplicity 1 result).

(3) If $f \in M_k$ and $T_m f = \lambda_m f \quad \forall m \geq 1$ and $f = \sum_{n=0}^{\infty} a_n q^n$ with $a_1 = 1$ then $a_m = \lambda_m$ and for such an eigenform we have

<u>Thm 6.9</u> Let $f = \sum_{n=0}^{\infty} c_n q^n$ be an eigenfunction of all Hecke operators T_m with $c_1 = 1$. Then

(1) $c_{mn} = c_n c_m$ if $(m,n) = 1$

(2) $c_{p^r} c_p = c_{p^{r+1}} + p^{k-1} c_{p^{r-1}}$

<u>Proof</u> It is easy to see λ_n satisfy the same properties as T_n from Thm 6.5 and hence Cor 6.7. Since $\lambda_n = c_n$ thm 6.9 follows

∎

<u>Cor 6-10</u> $\Delta(z) = q\prod_{n=1}(1-q^n)^{24} = \sum \tau(n) q^n \in S_{12}(\Gamma)$

the discriminant function. Then

$\tau(nm) = \tau(n)\tau(m)$, $(m,n)=1$ and

$\tau(p^r)\tau(p) = \tau(p^{r+1}) + p^{11}\tau(p^{r-1})$

<u>Proof.</u> Since $\dim S_{12}(\Gamma) = 1$ and $T_m : S_k \to S_k$

Δ is an eigenfunction $\forall$ T_m i.e. $T_m\Delta = \lambda_m \Delta$.

Since $\tau(1) = 1$, $\lambda_m = \tau(m)$ and

the multip. properties of τ follow from

Thm 6.9.

$\boxtimes$

The two multip. properties ⓐ, and ⓑ in

Thm 6.9 can be put into an analytic

statement about the Dirichlet series

$L(f,s) = \sum_{n=1}^{\infty} c_n n^{-s}$. Namely

<u>Thm 6-11</u> Let $f(z) = \sum c_n q^n \in S_k$

be a normalized eigenfunction for all T_m

Then $L(f,s) = \sum_{n=1}^{\infty} c_n n^{-s}$

$$= \prod_p \left(1 - c_p p^{-s} + p^{k-1} p^{-2s}\right)^{-1}$$

$\mathrm{Re}\, s > k/2 + 1$

Proof. Since $c_{nm} = c_n c_m$ for $(n,m)=1$

$$\sum_{n=1}^{\infty} c_n n^{-s} = \prod_p \sum_{\ell=0}^{\infty} c_{p^\ell} p^{-\ell s} \qquad \text{Put } X = p^{-s}$$

fund. thm of arithmetic (Exercise 3 in serie 4)

w.t.s. $\displaystyle\sum_{\ell=0}^{\infty} c_{p^\ell} X^\ell = \left(1 - c_p X + p^{k-1} X^2\right)^{-1}$

let $\displaystyle G(X) = \left(1 - c_\ell X + p^{k-1} X^2\right)\left(\sum_{\ell=0}^{\infty} c_{p^\ell} X^\ell\right)$

w.t.s. $G(X) \equiv 1$.

Now

Coef of X in $G(X) = c_p - c_p = 0$

coef of $X^{\ell+1}$ in $G(X) = c_{p^{\ell+1}} - c_p c_{p^\ell} + p^{k-1} c_{p^{\ell-1}} \underset{(b)}{=} 0$.

Coef of $X^0 = c_1 = 1$ (Since f is normalized

Hence $G(X) \equiv 1$

▨

We will introduce 2 more operators

<u>Defn</u> (U_m und V_m - operators) Let $f(q) = \sum_{n=0}^{\infty} a_n q^n \in \mathbb{C}[[q]]$

$$U_m f(q) = \sum_{m|n}^{\infty} a_n q^{n/m} = \sum_{n=0}^{\infty} a_{mn} q^n$$

$$V_m f(q) = \sum_{n=0}^{\infty} a_n q^{mn} \qquad w/ \quad q = e^{2\pi i \tau}$$

$$(U_m f)(\tau) = \frac{1}{m} \sum_{j=0}^{m-1} f\left(\frac{\tau + j}{m}\right) \qquad (V_m f)(\tau) = f(m\tau)$$

<u>Prop</u>. We have the following equality of operators on $\mathcal{U}_k(\Gamma)$

$$T_m = \sum_{d|m} d^{k-1} V_d \circ U_{m/d}.$$

<u>Proof</u> $\quad T_m f = m^{k/2-1} \sum_{\substack{ad=m \\ b \bmod d}} f\Big|_k \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$

$$(T_m f)(\tau) = m^{\frac{k}{2}-1} \sum_{ad=m} \sum_{b=0}^{d-1} m^{k/2} d^{-k} f\left(\frac{a\tau + b}{d}\right)$$

$$= m^{k/2-1} \sum_{ad=m} m^{k/2} d^{-k+1} (V_a \circ U_d f)(\tau)$$

$$= \sum_{a|m} a^{k-1} (V_a \circ U_{m/a} f)(\tau)$$

$\boxtimes$.

Now we go back to the proof of

<u>Thm 6.5</u> For any $m, n \geq 1$ we have

$$T_m T_n = \sum_{d \mid (m,n)} d^{k-1} T_{\frac{mn}{d^2}}$$

<u>Proof</u>. We write formally for T_n

$$T_n = \frac{1}{n} \sum_{ad=n} a^k \sum_{b \bmod d} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

$$mn\, T_m T_n = \sum_{\substack{a_1 d_1 = m \\ a_2 d_2 = n}} (a_1 a_2)^k \sum_{\substack{b_1 \bmod d_1 \\ b_2 \bmod d_2}} \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix}$$

$$\begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 d_2 \\ 0 & d_1 d_2 \end{pmatrix}$$

$$= \begin{pmatrix} \delta & \\ & \delta \end{pmatrix} \begin{pmatrix} a_1' a_2 & b \\ & d_1 d_2' \end{pmatrix}$$

where $\delta = (a_1, d_2)$ so that $\delta a_1' = a_1$
$\delta d_2' = d_2$

and $(a_1', d_2') = 1$

and $b = a_1' b_2 + b_1 d_2'$

Note $\delta \mid m = a_1 d_1$ and $\delta \mid n = a_2 d_2$

Hence we can write

$$mn\, T_m T_n = \sum_{\delta \mid (m,n)} \delta^k \sum_{\substack{a_1 d_1 = m/\delta \\ a_2 d_2 = n/\delta \\ (a_1, d_2) = 1}} (a_1 a_2)^k \sum_{\substack{b_1 \bmod d_1 \\ b_2 \bmod d_2 \delta}} \begin{pmatrix} a_1 a_2 & b \\ & d_1 d_2 \end{pmatrix}$$

$b = a_1 b_2 + b_1 d_2$ (we called d_2', d_2 and a_1' : a_1)

Now given a_1, a_2, d_1, d_2 as above, the upper right entry b covers every class modulo $d_1 d_2$ exactly δ times, since

$b = a_1 b_2 + b_1 d_2$, $(a_1, d_2) = 1$ b_1 runs mod d_1, b_2 runs mod $d_2 \delta$

Hence

$$mn\, T_m T_n = \sum_{\substack{\delta \mid (m,n) \\ (a_1, d_2) = 1}} \delta^k \cdot \delta \sum_{\substack{a_1 d_1 = m/\delta \\ a_2 d_2 = n/\delta}} (a_1 a_2)^k \sum_{b \bmod d_1 d_2} \begin{pmatrix} a_1 a_2 & b \\ 0 & d_1 d_2 \end{pmatrix}$$

Put $a = a_1 a_2$ $d = d_1 d_2$ so that $ad = \frac{mn}{\delta^2}$

Conversely given a factorization $ad = \frac{mn}{\delta^2}$, $\exists$ unique factorizations $a = a_1 a_2$ $d = d_1 d_2$ with $(a_1, d_2) = 1$ $a_1 d_1 = m/\delta$ $a_2 d_2 = n/\delta$

Indeed take $a_1 = \frac{m}{(m, \delta d)}$ $d_2 = \frac{\delta d}{(m, \delta d)}$

and check that it works.

Hence we can write

$$mn\, T_m T_n = \sum_{\delta \mid (m,n)} \delta^{k+1} \sum_{ad = \frac{mn}{\delta^2}} \sum_{b \bmod d} \begin{pmatrix} a & b \\ & d \end{pmatrix}$$

But $$T_{\frac{mn}{\delta^2}} = \frac{1}{mn/\delta^2} \sum_{ad = \frac{mn}{\delta^2}} a^k \sum_{b \bmod d} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

Hence $$mn\, T_m T_n = mn \sum_{\delta \mid (m,n)} \delta^{k-1}\, T_{\frac{mn}{\delta^2}}$$ as wanted $\blacksquare$.